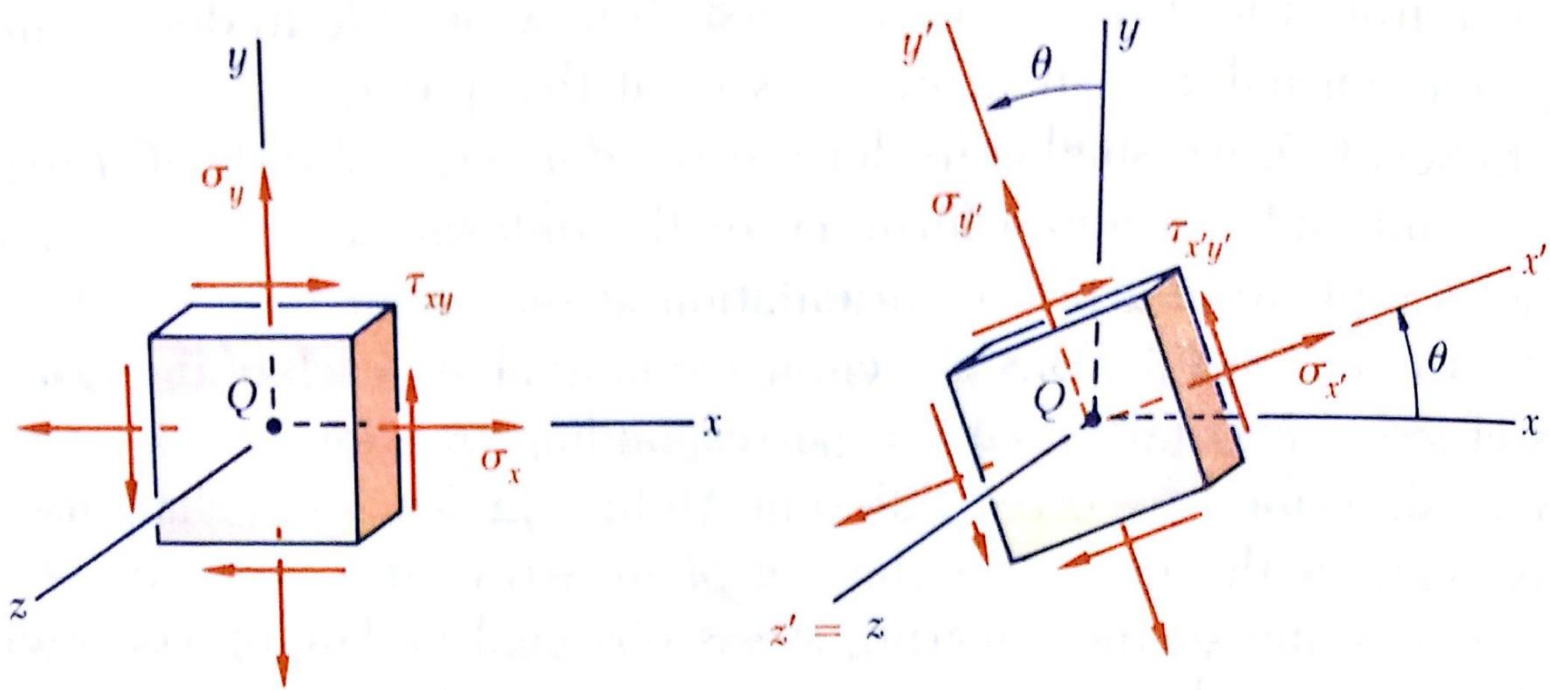
# Plasticity and Deformation Processes

Yielding criteria and the associated stress calculations

A state of plane stress exists at a point Q with $\sigma_z = \tau_{zx} = \tau_{zy} = 0$. The state of plane stress is defined by the stress components $\sigma_x, \sigma_y, \tau_{xy}$ associated with the material shown:




If the material is rotated through an angle $\theta$ about the z axis, the stress components change to $\sigma_x{}', \sigma_y{}', \tau_{xy}{}'$ which can be expressed in terms of $\sigma_x, \sigma_y, \tau_{xy}$ and $\theta$

Consider a prismatic element with faces respectively perpendicular to the x, y and x' axes:

If the area of the oblique face is ΔA, the areas of the vertical and horizontal faces are equal to ΔA cosθ, and ΔA sinθ respectively.

The mechanical equilibrium along the x' and y' axes require that

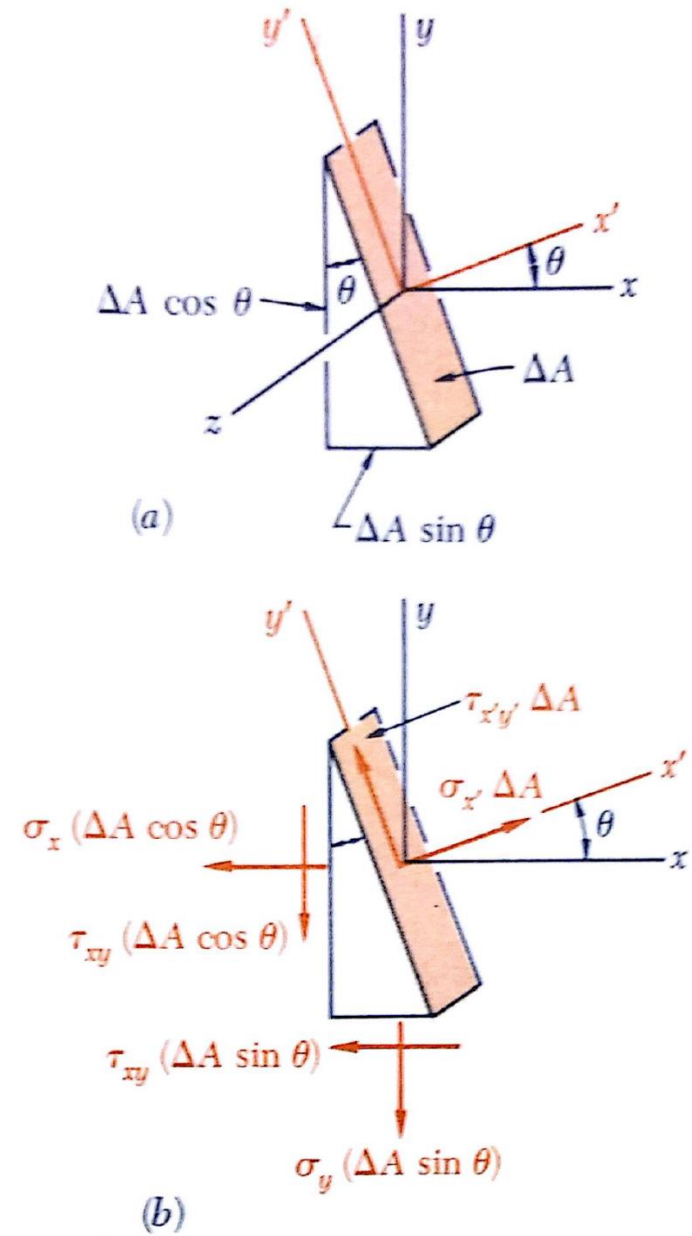
$$\sum F_{x'} = 0, \quad \sigma_x{}'\Delta A - \sigma_x(\Delta A\cos\theta)\cos\theta - \tau_{xy}(\Delta A\cos\theta)\sin\theta - \sigma_y(\Delta A\sin\theta)\sin\theta - \tau_{xy}(\Delta A\sin\theta)\cos\theta = 0$$

$$\sum F_{y'} = 0, \quad \tau_{x'y'}\Delta A + \sigma_x(\Delta A\cos\theta)\sin\theta - \tau_{xy}(\Delta A\cos\theta)\cos\theta - \sigma_y(\Delta A\sin\theta)\cos\theta + \tau_{xy}(\Delta A\sin\theta)\sin\theta = 0$$

The first equation is solved for $\sigma_x{}'$ and the second for $\tau_{x'y'}$ as

$$\sigma_x{}' = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\tau_{x'y'} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$



After simplifications using trigonometric substitutions we obtain the normal and shear stresses on the rotated material as

$$\sigma_x{}' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_y{}' = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

The expression for the normal stress $\sigma_y{}'$ is obtained by replacing θ by the angle θ+90 that the y' axis forms with the x axis.

Adding the two normal stresses we see that

$$\sigma_x{}' + \sigma_y{}' = \sigma_x + \sigma_y$$

In the case of plane stress, the sum of the normal stresses exerted on a cubic material is independent of the orientation of the material since $\sigma_z = \sigma_{z'} = 0$

The equations obtained for the normal and shear stresses in the rotated material under plane stress condition are the parametric equations of a circle

If we plot a point M in the rectangular axes with the coordinates $(\sigma_x{}', \tau_{x'y'})$ for any given value of the parameter θ, all the other possible points will lie on a circle.

$$\sigma_x{}' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

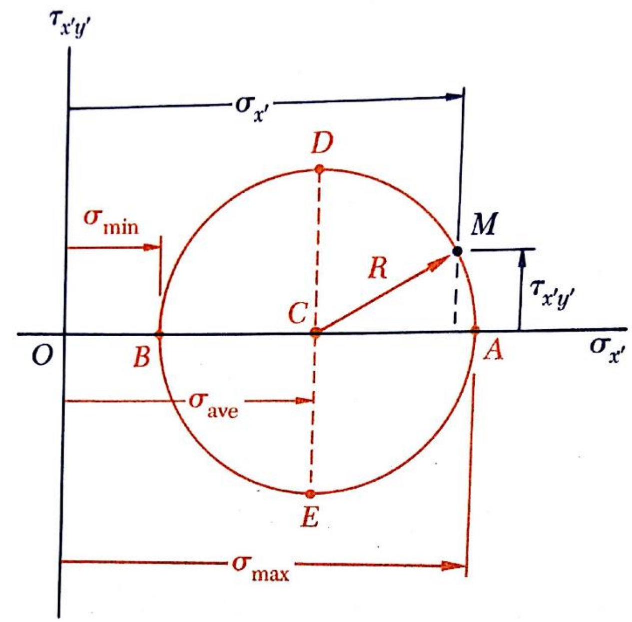
The angle θ in the equations can be eliminated by algebraic simplifications and addition of the two equations:

$$\left(\sigma_x{}' - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}{}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2$$

Where $\frac{\sigma_x+\sigma_y}{2} = \sigma_{ave}$ and $\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}{}^2 = R^2$

So $$(\sigma_x{}' - \sigma_{ave})^2 + \tau_{x'y'}{}^2 = R^2$$

Which is the equation of a circle of radius R centered at the point C of coordinates $(\sigma_{ave}, 0)$



The point A where the circle intersects the horizontal axis is the maximum value of the normal stress $\sigma_x{}'$ and the other intersection point B is the minimum value. Both points correspond to a zero value of shear stress $\tau_{x'y'}$. These are the principle stresses.

Since $\sigma_{max} = \sigma_{ave} + R$ $\sigma_{min} = \sigma_{ave} - R$

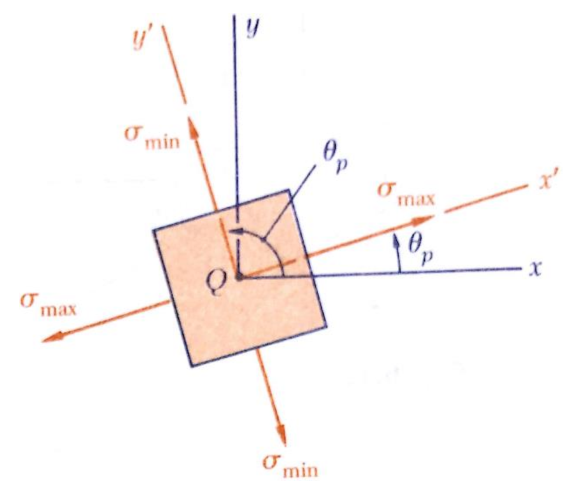
$$\sigma_{max,min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

The rotation angles that produce the principal stresses with no shear stress is obtained from the equation of shear stress

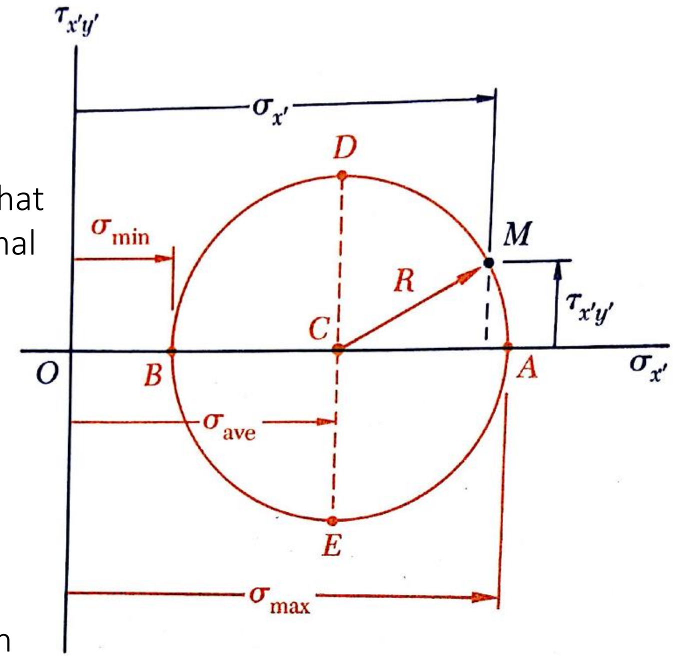
$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta = 0$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

This equation gives two $\theta_p$ values that are 90 apart. Either of them can be used to determine the orientation of the corresponding rotated plane. These planes are the principal planes of stress at point Q



The points D and E are located on the vertical diameter of the circle corresponding to the largest numerical value of the shear stress $\tau_{x'y'}$. These points have the same normal stresses of $\sigma_{ave}$. So the rotation that produces the maximum shear stresses can be obtained from the normal stress equations.

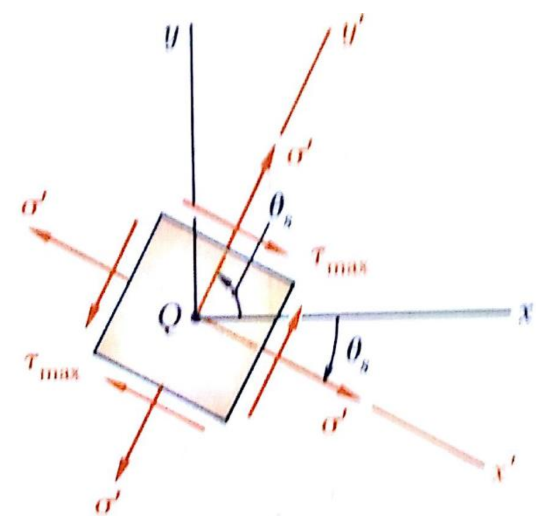



$$\sigma_x{}' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta = \frac{\sigma_x + \sigma_y}{2}$$

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

This equation gives two $\theta_s$ values that are 90 apart. Either of them can be used to determine the orientation of corresponding rotated plane that produces the maximum shear stress which is equal to

$$\tau_{max} = R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$



The normal stress corresponding to the condition of maximum shear stress is

$$\sigma_x{}' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

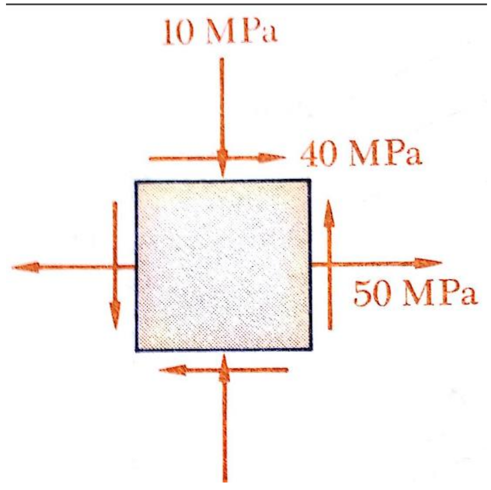
Also

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} = -\left(\tan 2\theta_p\right)^{-1} = -\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)^{-1}$$

This means that the angles $\theta_s$ and $\theta_p$ are 45 apart

So the planes of maximum shear stress are oriented at 45 to the principal planes

Example – Determine the principal planes, principle stresses, maximum shear stress and the corresponding normal stress for the state of plane stress shown



Yield criteria for ductile materials under plane stress

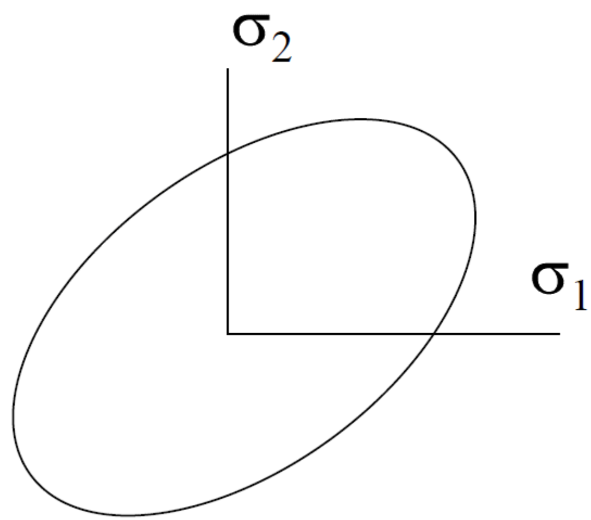
When a ductile material is under uniaxial stress, the value of the normal stress $\sigma_x$ which will cause the material to yield can be determined simply from a stress-strain diagram obtained by a tensile test.

The material will deform plastically when $\sigma_x > \sigma_{Yield}$

On the other hand when a material is in a state of multiaxial stress, the material will yield when the maximum value of the shear stress exceeds the corresponding value of the shear stress in a tensile-test specimen as it starts to yield.

Maximum shear stress criterion is based on the observation that yield in ductile materials is caused by slippage of the material along oblique surfaces and is due primarily to shear stresses.

In the plane stress condition the material can be represented as a point under principal stresses $\sigma_a, \sigma_b$



Recall that the maximum value of shear stress at a point under a centric axial load is equal to half the value of the corresponding normal axial stress.

Thus at yielding

$$\tau_{max} = \frac{1}{2}\sigma_Y$$

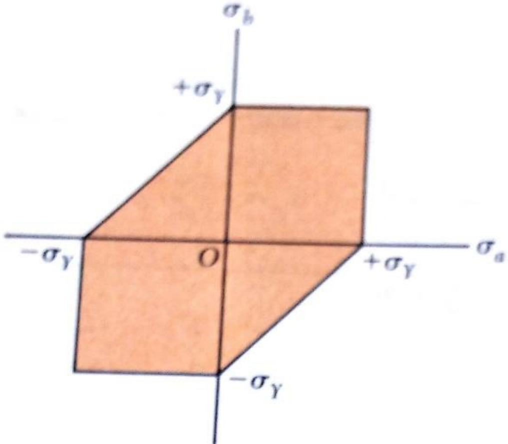
Also for plane stress condition if the principle stresses are both positive or both negative, the maximum value of the shear stress is equal to $\frac{1}{2}|\sigma_{max}|$

Therefore $\quad |\sigma_a| > \sigma_Y \quad$ or $\quad |\sigma_b| > \sigma_Y$

If the maximum stress is positive and the minimum stress negative, the maximum value of the shear stress is equal to $\frac{1}{2}(|\sigma_{max}| - |\sigma_{min}|)$

Therefore $\quad (|\sigma_a| - |\sigma_b|) > \sigma_Y$

These relations produce a hexagon in the xy plane, called Tresca's hexagon. Any given state of stress will be represented in the figure by a point.



Maximum distortion energy criterion is based on the determination of the distortion energy in a given material, which is the energy consumed by a change in the shape of the material.

Also called von Mises criterion, it states that a material will yield when the maximum value of the distortion energy per unit volume exceeds the distortion energy per unit volume required to cause yield in a tensile test specimen.

The distortion energy in an isotropic material under plane stress is

$$U_d = \frac{1}{6G}({\sigma_a}^2 - \sigma_a\sigma_b + {\sigma_b}^2)$$

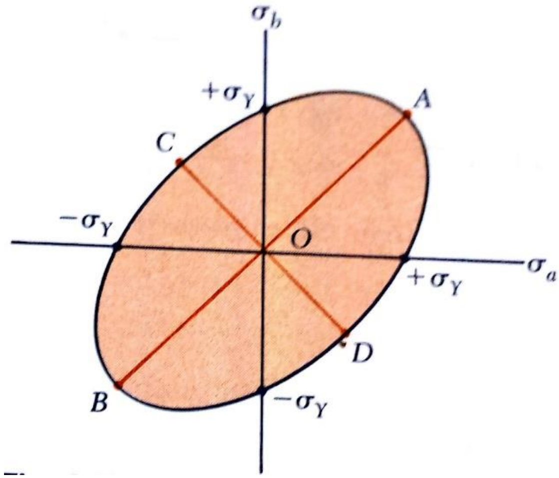
In the case of a tensile test specimen yielding at $\sigma_Y$

$$U_Y = \frac{1}{6G}({\sigma_Y}^2)$$

Thus the maximum distortion energy criterion indicates that the material yields when $U_d > U_Y$:

$${\sigma_a}^2 - \sigma_a\sigma_b + {\sigma_b}^2 > {\sigma_Y}^2$$

This equation produces an ellipse in the principal stress plane



The von Mises yield criterion is given by

$$\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}=2\sigma_y$$

Or

$$\sqrt{\left(\sigma_x-\sigma_y\right)^2+\left(\sigma_y-\sigma_z\right)^2+(\sigma_z-\sigma_x)^2+6\left(\tau_{yz}{}^2+\tau_{zx}{}^2+\tau_{xy}{}^2\right)}=2\sigma_y$$

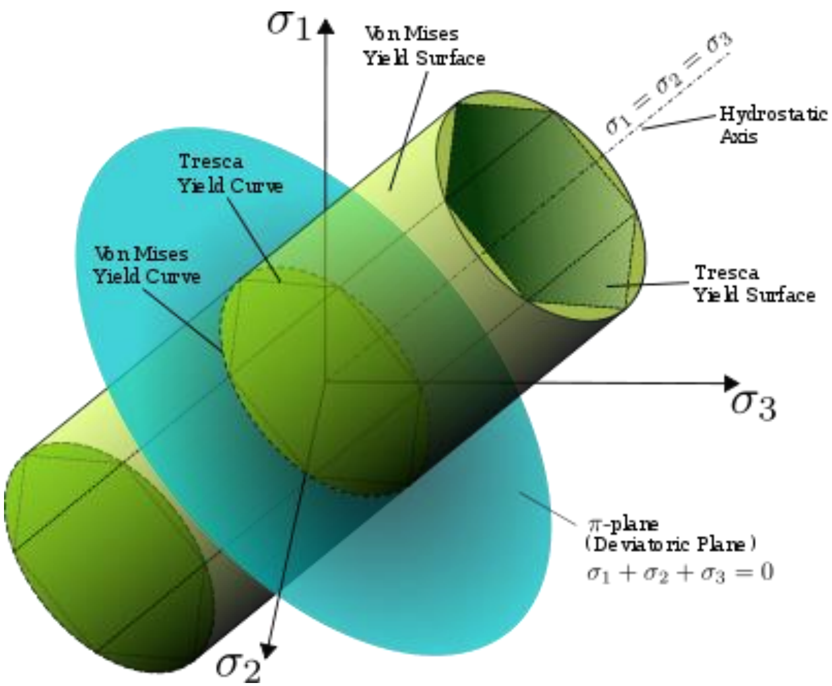
In terms of effective stress the criterion is

$$\sigma_{eff}=\frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}=\sigma_y$$

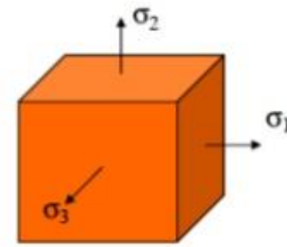
$$\sigma_{eff}=\frac{\sqrt{2}}{2}\sqrt{\left(\sigma_x-\sigma_y\right)^2+\left(\sigma_y-\sigma_z\right)^2+(\sigma_z-\sigma_x)^2+6\left(\tau_{yz}{}^2+\tau_{zx}{}^2+\tau_{xy}{}^2\right)}=\sigma_y$$

For plane states of stress, the yield condition is the interaction of the cylinder with the principal stress plane, which is a yield ellipse

The von Mises yield criterion is visualized as a circular cylinder in the stress space




✓ **Body subjected to principal stresses :**




$$U = \frac{1}{2}(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3)$$

$$U = 1/2E\,[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\,\nu\,(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)]$$

✓ **For the onset of yielding :**

$$Y^2/2E = 1/2E\,[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\,\nu\,(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)]$$

✓ **Yield function**

$$f = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \upsilon(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) - Y^2$$

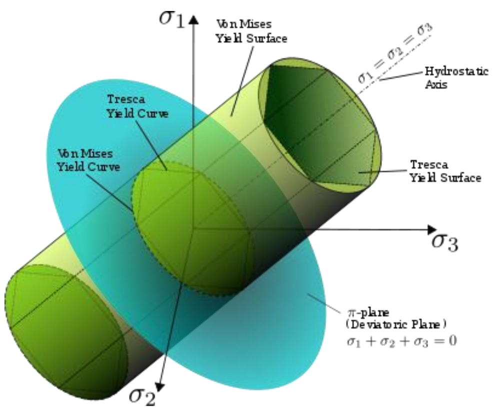
$$f = \sigma_e^2 - Y^2$$

$$\boxed{\text{Yielding} \Rightarrow f = 0,\ \text{safe } f < 0}$$

The axis of the cylinder passes through the origin of the coordinates for unyielded material

It is inclined equal amount to the three axes and represents pure hydrostatic stress for elastic deformations.

The effective stress is the uniaxial stress that is equally distant from the yield surface or located on it




The effective stress or the stress intensity for an elastic material is expressed as

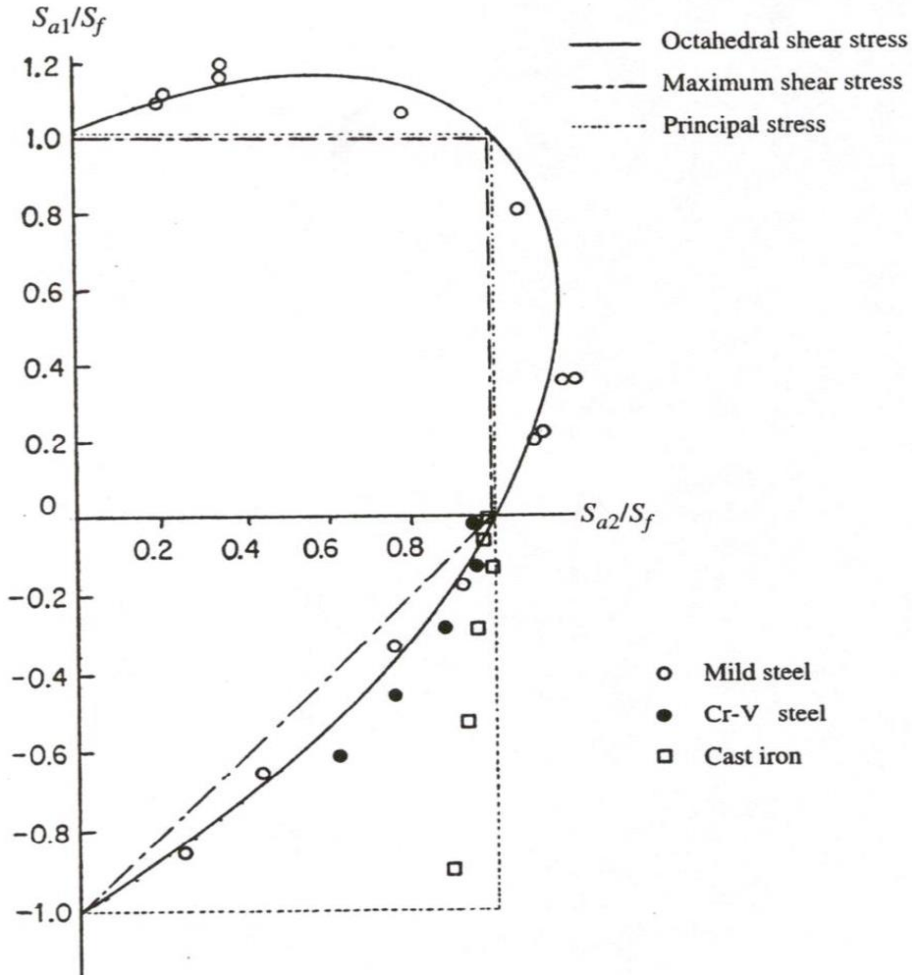
$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2 + 6\left({\tau_{yz}}^2 + {\tau_{zx}}^2 + {\tau_{xy}}^2\right)}$$

And the effective strain as

$$\varepsilon_{eff} = \frac{\sqrt{2}}{2(1+\nu)}\sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + \left(\varepsilon_y - \varepsilon_z\right)^2 + (\varepsilon_z - \varepsilon_x)^2 + \frac{3}{2}\left({\gamma_{yz}}^2 + {\gamma_{zx}}^2 + {\gamma_{xy}}^2\right)}$$

And $\sigma_{eff} = E\varepsilon_{eff}$

Yield criteria for deformation of metals under plane stress




The data for the mild steel and Cr-V steel which behave in a ductile manner agree well with the octahedral shear stress (von Mises) criterion

Data for cast iron which behaves in a brittle manner, agrees better with the maximum principal stress criterion:

$$\sigma_1 = \sigma_y$$

Brittle materials fail suddenly in a tensile test by rupture without any prior yielding.

When a brittle material is under uniaxial tensile stress, the value of the normal stress which causes it to fail is equal to the ultimate strength of the material as determined from a tensile test.

When a brittle material is under plane stress, the principal stresses are compared to the ultimate strength obtained from the uniaxial tensile test.

Maximum principal stress criterion states that a brittle material will fail when the maximum normal stress exceeds the ultimate strength obtained from the uniaxial tensile test:

$$|\sigma_a| > \sigma_U \quad \text{or} \qquad |\sigma_b| > \sigma_U$$

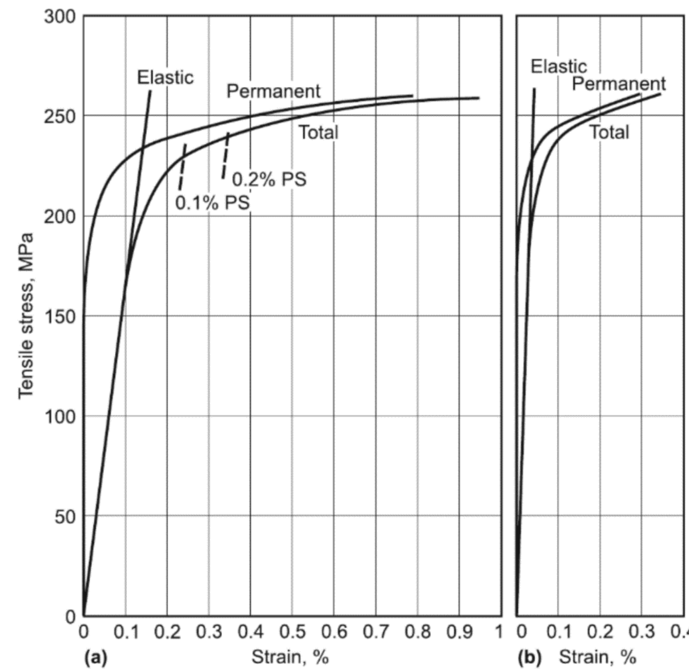
This criterion forms a square area centered on the xy plane. The criterion is based on the assumption that the ultimate strength of materials under tension and compression are equal, which is an overestimation for most materials as the presence of cracks and flaws often weaken the material under tension

Example – Evaluate the yielding stress condition for a ductile cast iron using maximum shear stress, maximum principal stress and maximum distortion energy criteria.

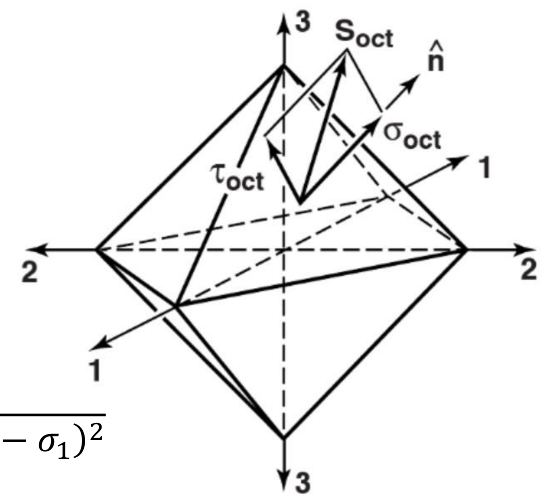
$|\sigma_a| > \sigma_Y$ or $|\sigma_b| > \sigma_Y$ or $(|\sigma_a| - |\sigma_b|) > \sigma_Y$

$|\sigma_a| > \sigma_U$ or $|\sigma_b| > \sigma_U$

${\sigma_a}^2 - \sigma_a\sigma_b + {\sigma_b}^2 > {\sigma_Y}^2$



Prediction of yielding under multiaxial loading according to the maximum shear stress criterion involves the analysis of the octahedral planes




There are eight octahedral planes making equal angles with the principal stress directions
The shearing stress on these planes is given by

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

Or with nonprinciple stresses:

$$\tau_{oct} = \frac{1}{3}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{yz}{}^2 + \tau_{zx}{}^2 + \tau_{xy}{}^2\right)}$$

The shear strain acting on an octahedral plane is given by

$$\gamma_{oct} = \frac{2}{3}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2}$$

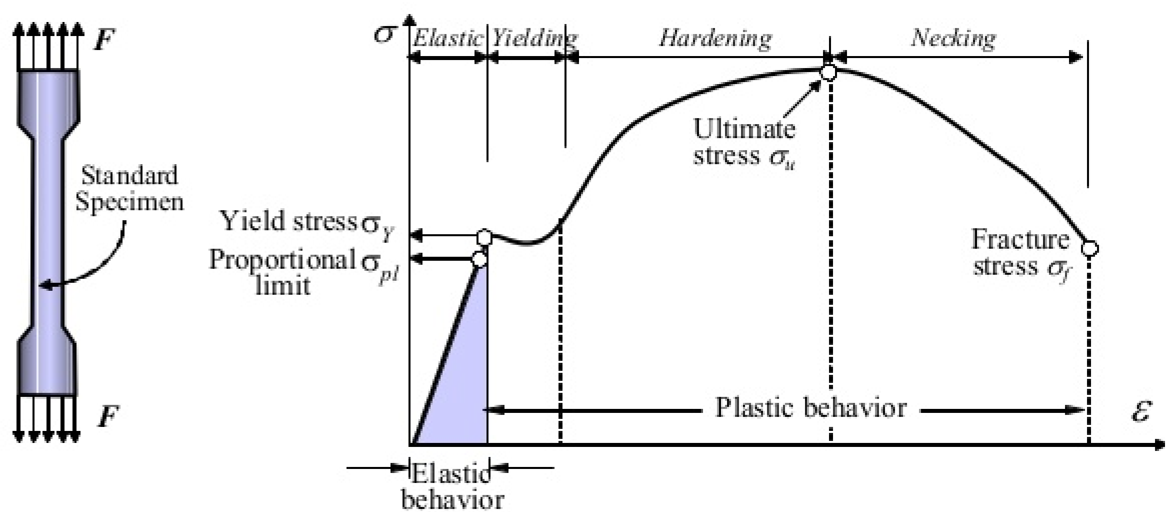
Or

$$\gamma_{oct} = \frac{2}{3}\sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + \left(\varepsilon_y - \varepsilon_z\right)^2 + (\varepsilon_z - \varepsilon_x)^2 + \frac{3}{2}\left(\gamma_{yz}{}^2 + \gamma_{zx}{}^2 + \gamma_{xy}{}^2\right)}$$

Materials behave elastically until the deforming force increases beyond the yield stress. At that point, the material is irreversibly and permenantly deformed.

Irreversible deformation at normal temperatures cause the dislocations to accumulate, interact with one another, and serve as pinning points or obstacles that significantly impede their motion. This leads to an increase in the yield strength of the material and a subsequent decrease in ductility.

Its extent is dependent on the material and the dislocation density




Because dislocation motion is hindered, plastic deformation cannot occur at normal stresses. The yield stress increases as a result.
At a stress lower than the yield stress, a cold-worked material will continue to deform using the only mechanism available: elastic deformation and the modulus of elasticity is unchanged. With increasing stress the strain-field interactions are overcome and plastic deformation resumes. It has now become a brittle material. If dislocation motion and plastic deformation have been hindered enough by dislocation accumulation, and stretching of electronic bonds and elastic deformation have reached their limit, a third mode of deformation occurs: fracture.

Increase in the number of dislocations is a quantification of work hardening. Plastic deformation occurs as a consequence of work being done on and energy added to a material. In addition, the energy is almost always applied fast enough and in large enough magnitude to not only move existing dislocations, but also to produce a great number of new dislocations.

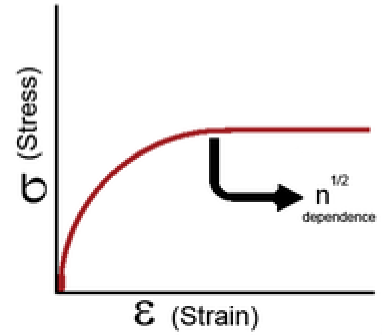
$$\Delta\tau \;= Gb\rho^{1/2}$$

Work hardening has a half root dependency on the number of dislocations. The material exhibits high strength if there are either high levels of dislocations (greater than $10^{14}$ dislocations per $m^2$) or no dislocations. A moderate number of dislocations (between $10^7$ and $10^9$ dislocations per $m^2$) typically results in low strength

Work hardening phenomenon is formulated as a power law relationship between the stress and the amount of plastic strain:

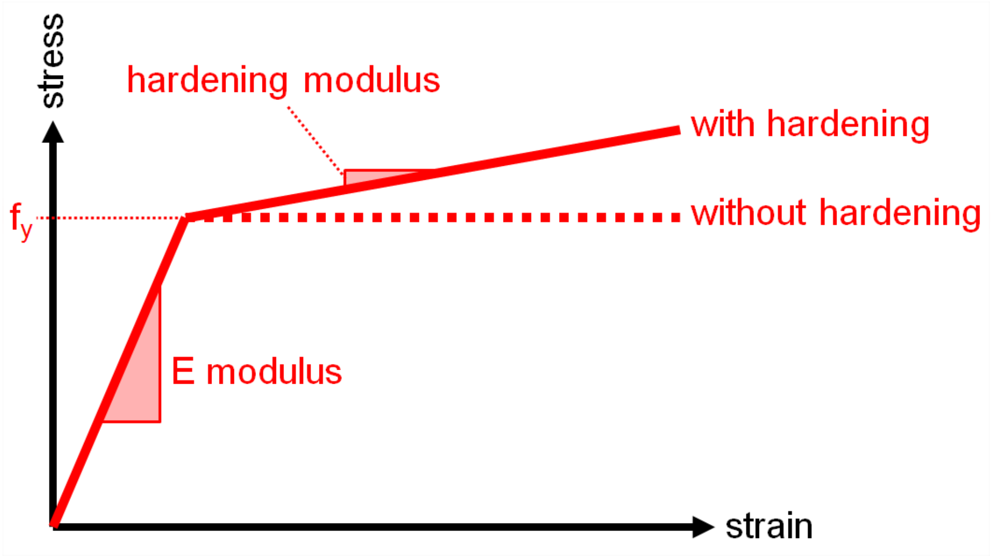
$$\sigma = K\epsilon_p^n \quad \text{or} \quad \sigma = \sigma_y + K\epsilon_p^n$$

where σ is the stress, $\sigma_y$ is the yield stress, K is the strength index or strength coefficient, $\varepsilon_p$ is the plastic strain and n is the strain hardening exponent.



Elastoplastic material is a model for easily plastically deformed materials that simplifies the calculations significantly

In reality all materials harden to an extent and mostly non-linearly without a definite hardening modulus




Calculation of the plastic deformations similarly to elastic deformations using Hooke’s law is only possible by using a dynamic modulus which is a definite function of applied strain

Deformation theory helps us do that once we determine the yielding condition and the effective stress state